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Brief communication

Infinitesimally null Ricci isotropic Lorentz manifolds

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Abstract

We introduce the concept of infinitesimal null Ricci isotropy for Lorentz manifolds as a generalization of the infinitesimal null isotropy. We give a criterion for infinitesimally isotropic pseudo-Riemannian manifolds to be conformally flat.

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1. Infinitesimal null Ricci isotropy

A characterization of Robertson–Walker metrics was first obtained by Karcher [3] by introducing the concept of the infinitesimal spatial isotropy, which means that for a fixed timelike vector field U, the spacelike sectional curvatures for planes in $U^\perp$ and the timelike sectional curvatures for planes containing U are functions on a Lorentz manifold M. On the other hand, it can be shown that this notion is equivalent to the infinitesimal null isotropy, which means that the null sectional curvature restricted on the null congruence of U is a function on M (cf. [2,4]).

As a natural extension of the infinitesimal null isotropy, we consider the question, what happens if the null Ricci curvature restricted on the null congruence of U is a function on M. One can easily check the condition that the null Ricci curvature restricted on the null congruence of U is a function on M is equivalent to the following definition.

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Definition 1. Let (M, g) be an n-dimensional Lorentz manifold and U be a timelike unit vector field on M. We say that M is *infinitesimal null Ricci isotropic* with respect to U if there are functions A and B on M such that the restriction of the Ricci tensor to the null vector bundle T^0M can be expressed as

$$Ric|_{T^0M} = A\, U^* \otimes U^* + Bg,$$

where U^* is the one-form metrically equivalent to U.

It is quite clear that M with a perfect fluid is necessarily infinitesimally null Ricci isotropic. The following theorem shows that the infinitesimal null Ricci isotropy is also a sufficient condition for a perfect fluid.

Theorem 2. *Let M be a four-dimensional Lorentz manifold, and let U be a timelike unit vector field on M. Then M is infinitesimally null Ricci isotropic with respect to U if and only if M has a perfect fluid whose flow vector field is U.*

Proof. From a direct computation, we get the scalar curvature $S = -A + 4B$, the Ricci curvatures $Ric(U, U) = A - B$, $Ric(U, X) = 0$, and $Ric(X, Y) = Bg(X, Y)$, for $X, Y \in U^{\perp}$. Thus from the Einstein equation, we can obtain the following identities:

$$\mathbf{T}(U, U) = \frac{1}{8\pi}\left(\frac{A}{2} + B\right), \quad \mathbf{T}(U, X) = 0, \quad \mathbf{T}(X, Y) = \frac{1}{8\pi}\left(\frac{A}{2} - B\right) g(X, Y),$$

for $X,\ Y \in U^{\perp}$. □

We remark that by Theorem 2, if the restriction of the stress–energy tensor T of a space–time M to the null vector bundle T^0M is of the form

$$\mathbf{T}|_{T^0M} = (\rho + \mathrm{p})\, U^* \otimes U^* + \mathrm{p}g,$$

then M has a perfect fluid whose flow vector field is U.

It is well known that the infinitesimally null isotropic space–times are exactly conformally flat space–times with a perfect fluid (cf. [4]). Thus, by Theorem 2, the infinitesimal null isotropy implies the infinitesimal null Ricci isotropy, but not conversely in general.

Theorem 3. *Let M be a four-dimensional Lorentz manifold and U a timelike vector field on M. Then M is infinitesimally null isotropic with respect to U if and only if M is infinitesimally null Ricci isotropic with respect to U, and the sectional curvature of the planes containing U is constant at each point p in M.*

Proof. The "only if" part is clear by Lemma and Theorem 1 in [4]. Suppose that M is infinitesimally null Ricci isotropic with respect to U and the sectional curvature of the plane containing U is constant—$\mu(p)$ at each point p in M. Then by a direct computation, we have

$$K(x, y) + K(x, z) = B(p) + \mu(p)$$

for orthonormal vectors x, y, z in $U_p^{\perp} \subseteq T_pM$, $p \in M$. Since $\dim(M) = 4$, we have $K(x, y) = \frac{1}{2}(B(p) + \mu(p))$ for any linearly independent vectors x, y in T_pM perpendicular to U_p. Thus M is infinitesimally null isotropic with respect to U. □

Note that the constancy of the sectional curvatures of "the planes containing U" in Theorem 3 can be replaced by constancy of the sectional curvatures of "the planes perpendicular to U".

2. Infinitesimal isotropy and conformal flatness

Recently, García-Río and Küpeli [1] extended the concepts of those infinitesimal versions of null and spatial isotropies to the null isotropy and the infinitesimal isotropy for pseudo-Riemannian manifolds. They showed that a pseudo-Riemannian manifold M with indefinite metric is null isotropic at $p \in M$ if and only if M is conformally flat at p.

A pseudo-Riemannian manifold M is called *infinitesimally isotropic* with respect to the orthogonal decomposition $T_pM = W_1 \oplus W_2$ at $p \in M$ if

(a) there is $\mu \in \mathbb{R}$ such that $R(z, x)y = \mu g(x, y)z$ for all $z \in W_i$, $x, y \in W_j$ for $i \neq j$;

(b) there are $\kappa_i \in \mathbb{R}$ such that $R(x, y)z = \kappa_i R_0(x, y)z$ for all $x, y, z \in W_i$ for $i = 1, 2$, where R is the Riemannian curvature tensor and $R_0(x, y)z = g(y, z)x - g(x, z)y$. In particular, if $\dim(W_i) = 1$, then $\kappa_i = 0$.

García-Río and Küpeli [1] showed that if an indefinite pseudo-Riemannian manifold M whose Ricci tensor can be written as $Ric = \rho g_1 \oplus \lambda g_2$ for some $\rho, \lambda \in \mathbb{R}$ is null isotropic at $p \in M$, then M is infinitesimally isotropic with respect to $T_pM = W_1 \oplus W_2$, where $g_1 = g|_{W_1}$ and $g_2 = g|_{W_2}$. They also proved the converse for the special case: if $\dim(W_1) = 1$ (or $\dim(W_2) = 1$), then M is infinitesimally isotropic with respect to $T_pM = W_1 \oplus W_2$ if and only if M is null isotropic at $p \in M$ and $Ric = \rho g_1 \oplus \lambda g_2$.

The following theorem gives criteria for an infinitesimally isotropic pseudo-Riemannian manifold to be conformally flat.

Theorem 4. *Let M be a pseudo-Riemannian manifold of dimension $n \geq 4$ and $T_pM = W_1 \oplus W_2$, where W_1 and W_2 are orthogonal. Suppose that M is infinitesimally isotropic with respect to the orthogonal decomposition $T_pM = W_1 \oplus W_2$. M is conformally flat at p if and only if M satisfies either of the following:*

(i) $\dim(W_1) = 1$ *or* $\dim(W_2) = 1$,

(ii) $\kappa_1 + \kappa_2 = 2\mu$.

Proof. With a straightforward computation of the Weyl tensor W, we have

$$W(x, y, x, y) = \frac{n_2(n_2 - 1)}{(n-1)(n-2)}(\kappa_1 + \kappa_2 - 2\mu)F_0(x, y, x, y) \qquad \text{for } x, y \in W_1,$$

$$W(x, y, x, y) = \frac{n_1(n_1 - 1)}{(n-1)(n-2)}(\kappa_1 + \kappa_2 - 2\mu)F_0(x, y, x, y) \qquad \text{for } x, y \in W_2,$$

$$W(x,y,x,y) = -\frac{(n_1-1)(n_2-1)}{(n-1)(n-2)}(\kappa_1+\kappa_2-2\mu)F_0(x,y,x,y)$$
$$\text{for } x \in W_1, \quad y \in W_2,$$

where $n_1 = \dim(W_1)$, $n_2 = \dim(W_2)$ and $F_0(x,y,z,w) = g(x, R_0(z,w)y)$.

Suppose that $\dim(W_1) = 1$ (or $\dim(W_2) = 1$) or $\kappa_1 + \kappa_2 = 2\mu$. Then $W(x,y,x,y) = 0$ for $x, y \in W_1$, $W(x,y,x,y) = 0$ for $x, y \in W_2$, and $W(x,y,x,y) = 0$ for $x \in W_1$ and $y \in W_2$. By the symmetries of the Weyl tensor, $W = 0$ at p.

Suppose that M is conformally flat at p. If $\dim(W_1) \neq 1$ and $\dim(W_2) \neq 1$, then $\kappa_1 + \kappa_2 = 2\mu$ since $F_0 \not\equiv 0$. □

The Robertson–Walker space–times are infinitesimally null Ricci isotropic (i.e., perfect fluids) and conformally flat. In general, the locally symmetric spaces are Ricci parallel, but not conversely. However, one can easily prove that for a conformally flat pseudo-Riemannian manifold of dimension $n \geq 4$, Ricci parallel spaces are locally symmetric. Therefore, by solving $\nabla R = 0$ for these space–times,we can easily obtain the following classification: the Ricci parallel Robertson–Walker space–times are only open submanifolds of $\mathbb{R}^4_1$, $\mathbb{S}^4_1(\rho)$, $\mathbb{H}^4_1(\rho)$, $\mathbb{R} \times \mathbb{S}^3(\rho)$, $\mathbb{R} \times \mathbb{H}^3(\rho)$.

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